Math 53, Discussions 116 and 118 **2022-09-07**

Areas in polar coordinates

Answers included

Questions

Question 1. Find the slope of the tangent line to the polar curve $r = 1/\theta$ when $\theta = \pi$.

Question 2. Sketch the polar curve $r = 2 + \cos(\theta/3)$. Then set up an integral which evaluates the area of the innermost loop.

Question 3. Sketch the polar curve $r = \sin(3\theta)$. How many "petals" does it have? Set up an integral which computes the area of one petal.

Then answer the same questions for $r = \sin(4\theta)$.

More problems

Problem 1. Find the area of the region *underneath* the polar curve $r = \theta$, $2\pi/3 \le \theta \le 5\pi/6$, depicted in Figure 1, in two ways:

(a) Convert to parametric equations and use methods of §10.2.

(b) First compute the area of the region with corners O, B, and D using methods of §10.4. Then use that to find the desired area. **Hint:** Think about the right triangles $\triangle$BAO and $\triangle$DCO.

Check that you get the same answer. Which method do you find easier?

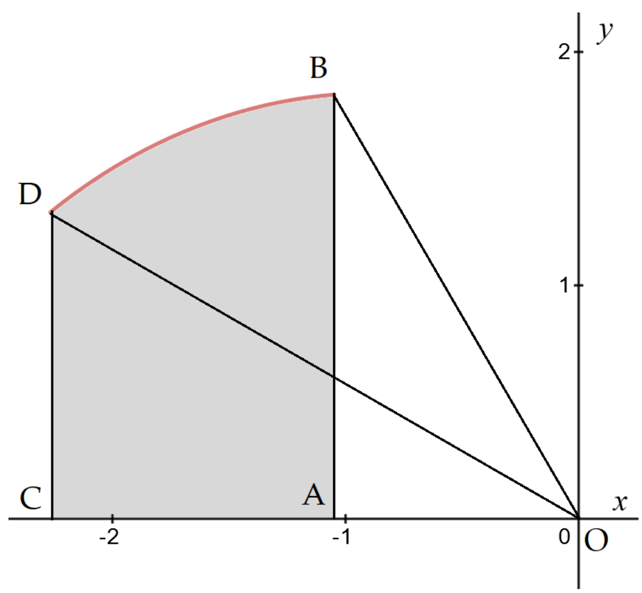

FIGURE 1

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. Rewriting the given polar curve as a parametric curve yields

$$x = \frac{1}{\theta}\cos\theta, \qquad y = \frac{1}{\theta}\sin\theta$$

and then the formula for slope gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{1}{\theta^2}\sin\theta + \frac{1}{\theta}\cos\theta}{-\frac{1}{\theta^2}\cos\theta - \frac{1}{\theta}\sin\theta}.$$

After we plug in $\theta = \pi$ we get the final answer $\boxed{-\pi}$.

Question 2. I drew a picture of this in class. The main point is to understand that the bounds of integration ought to be from 2π to 4π to cover the innermost loop, so our integral is

$$\int_{2\pi}^{4\pi} \frac{1}{2}(2+\cos(\theta/3))^2\,\mathrm{d}\theta.$$

Question 3. I drew pictures for these in class as well. The polar curve $r = \sin(3\theta)$ has 3 petals (and is traced out once in the interval $0 \le \theta \le \pi$), and the area of one is e.g.

$$\int_0^{\pi/3} \frac{1}{2}(\sin(3\theta))^2\,\mathrm{d}\theta.$$

The polar curve $r = \sin(4\theta)$ has 8 petals (and is traced out once in the interval $0 \le \theta \le 2\pi$) and the area of one is e.g.

$$\int_0^{\pi/4} \frac{1}{2}(\sin(4\theta))^2\,\mathrm{d}\theta.$$

Problem 1.

(a) The parametric equations for the curve are

$$x = \theta\cos\theta, \qquad y = \theta\sin\theta.$$

So the area in question is computed by the integral

$$\int_{5\pi/6}^{2\pi/3} (\theta\sin\theta)(\cos\theta - \theta\sin\theta)\,\mathrm{d}\theta.$$

Note that the bounds of integration are from $5\pi/6$ to $2\pi/3$ instead of vice versa (why?). This is a rather annoying integral, but not an impossible one; you can break it up and use integration by parts for example. The final answer is $\boxed{\pi^2\sqrt{3}/32 + 61\pi^3/1296}$.

(b) The area of the region with vertices OBD is given by

$$\int_{2\pi/3}^{5\pi/6} \frac{1}{2}\theta^2\,\mathrm{d}\theta = \frac{61\pi^3}{1296}.$$

The point D has (x, y) coordinates $(\frac{5\pi}{6}\cos(5\pi/6), \frac{5\pi}{6}\sin(5\pi/6))$. This means that the area of the right triangle OCD is

$$\frac{1}{2}(-\frac{5\pi}{6}\cos(5\pi/6))\frac{5\pi}{6}\sin(5\pi/6)) = \frac{25\pi^2}{96\sqrt{3}}.$$

Similarly one can see that the area of right triangle OAB is

$$\frac{1}{2}(-\frac{2\pi}{3}\cos(2\pi/3))\frac{2\pi}{3}\sin(2\pi/3) = \frac{\pi^2}{6\sqrt{3}}.$$

By examining the picture, one sees that the desired area can be computed as

$$\frac{61\pi^3}{1296} - \frac{\pi^2}{6\sqrt{3}} + \frac{25\pi^2}{96\sqrt{3}} = \boxed{\frac{\pi^2\sqrt{3}}{32} + \frac{61\pi^3}{1296}}.$$

Although solution (a) looks shorter in this write-up, that's only because I skipped the integral computation entirely! If it were included, it would be much longer...